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On the Asymptotic Behavior of Some Population Models, II

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0. INTRODUCTION

In a recent paper [2] we have proved several results about the asymptotic behavior of positive solutions of the Lotka–Volterra system

$$u_i' = u_i \left[b_i(t) - \sum_{j=1}^{n} a_{ij}(t) u_j \right], \qquad 1 \leq i \leq n, \tag{0.1}$$

where $n \geq 2$ and a_{ij}, b_i: $\mathbb{R} \to \mathbb{R}$ are continuous functions.

In [2] we assumed that the following hypotheses hold:

(H_1) a_{ij}, b_i are bounded for all $1 \leq i, j \leq n$.

(H_2) There exist positive constants $c_1, ..., c_n, m$ such that,

$$c_i a_{ii}(t) \geq m + \sum_{j \in J_i} c_j |a_{ji}(t)| \tag{0.2}$$

For all $1 \leq i \leq n$ and $t \in \mathbb{R}$ where, $J_i := \{1, ..., i-1, i+1, ..., n\}$, for $i = 1, ..., n$. Notice that (0.2) implies $\inf\{a_{ii}(t)\colon 1 \leq i \leq n, t \in \mathbb{R}\} > 0$.

(H_3) System (0.1) has a positive solution defined and bounded on $[0, \infty)$.

In this paper, we shall show that (H_1)–(H_2) imply (H_3). In fact, we shall prove that, if $u = (u_1, ..., u_n)$ is a solution to (0.1) and $u(\tau) > 0$ for some

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τ, then u is defined and bounded on $[\tau, \infty)$. From this and the results in [2], we obtain the following corollaries:

(C_1) If u, v are positive solutions to (0.1) then, $u(t) - v(t) \to 0$ as $t \to +\infty$.

(C_2) If a_{ij}, b_i are T-periodic (some $T > 0$) then (0.1) has a (unique and nonnegative) T-periodic solution u^0 such that

$$u(t) - u^0(t) \to 0 \qquad \text{as } t \to +\infty \tag{0.3}$$

for any positive solution u to (0.1).

(C_3) If a_{ij}, b_i are almost periodic and (0.1) has a positive solution $v = (v_1, ..., v_n)$ such that, $\liminf_{t\to+\infty} v_i(t) > 0$, for all i, then (0.1) has a (unique) almost periodic solution u^0 whose components are bounded below by positive constants.

The proof of our main result (Theorem 3.1) is based on the following theorem about the continuation of positive solutions of (0.1).

0.1. Theorem. *Assume $a_{ii}(t) > 0$ for $1 \le i \le n$ and $t \in \mathbb{R}$. Let us define $b_{ii}(t) = 0$ and $b_{ij}(t) = |a_{ij}(t)|/a_{ii}(t)$ for $i \neq j$, and suppose that the spectral radius of the matrix $(b_{ij}(t))$ is less than one for all $t \in \mathbb{R}$. If u is a solution of* (0.1) *and $u(\tau) > 0$ for some τ then u is defined on $[\tau, \infty)$.*

For facultative mutalism we obtain the following result.

0.2. Theorem. *Assume that a_{ij}, b_i are almost periodic and suppose that*:

(a) $a_{ij} \le 0$ *for $i \neq j$.*

(b) $\operatorname{Inf}\{b_i(t)\colon t \in \mathbb{R}, 1 \le i \le n\} > 0$.

If (0.2) holds, then (0.1) has an almost periodic solution u^0 whose components are bounded below by positive constants. Moreover, (0.3) holds for any positive solution u to (0.1).

Finally, we shall apply Corollaries (C_1)–(C_3) to prove some results about extinction.

1. Nonnegative Matrices

In this section, $A = (\alpha_{ij})$ denotes a real $n \times n$ matrix such that,

$$\alpha_{ij} \ge 0 \qquad \text{and} \qquad \alpha_{ii} > 0 \qquad \text{for } i, j = 1, ..., n. \tag{1.1}$$

We define $n \times n$ matrices $B = (\beta_{ij})$, $C = (\gamma_{ij})$ by: $\beta_{ii} = \gamma_{ii} = 0$, for $i = 1, ..., n$, and $\beta_{ij} = \alpha_{ij}/\alpha_{ii}$ and $\gamma_{ij} = \alpha_{ji}/\alpha_{ii}$ for $i \neq j$.

1.1. THEOREM. *The spectral radius $\nu(B)$ of B is less than one iff there exist positive constants $c_1, ..., c_n$ such that,*

$$c_i\alpha_{ii} > \sum_{j\in J_i} \alpha_{ij}c_j; \qquad 1 \le i \le n. \tag{1.2}$$

Proof. Assume that (1.2) is satisfied and let $\mu \in \mathbb{C}$ be an eigenvalue of B. Then, there exist $z_1, ..., z_n \in \mathbb{C}$ such that $|z_1| + \cdots + |z_n| > 0$ and,

$$\mu z_i = \sum_{j=1}^{n} \beta_{ij}z_j = \sum_{j\in J_i} \beta_{ij}z_j; \qquad 1 \le i \le n. \tag{1.3}$$

Now, let us fix $1 \le i \le n$ such that,

$$|z_i|/c_i \ge |z_j|/c_j \qquad \text{for } j = 1, ..., n. \tag{1.4}$$

From (1.3)–(1.4),

$$\begin{aligned} |\mu|\,|z_i|^2 = \left|\sum_{j\in J_i} \beta_{ij}z_j\bar{z}_i\right| &\le \sum_{j\in J_i} \beta_{ij}|z_j|\,|z_i| \\ &\le \sum_{j\in J_i} \beta_{ij}|z_i|^2 c_j/c_i, \end{aligned}$$

and hence, $|\mu| \le \sum_{j\in J_i} \alpha_{ij}c_j/(\alpha_{ii}c_i) < 1$, since (1.2) is satisfied. Thus, $\nu(B) < 1$.

To show the converse, we use induction on n. Notice that, for $n = 1$, there is nothing to prove. By a suitable application of the Perron–Frobenius theorem [1], there exist $\lambda \in [0, 1)$ and $b = \mathrm{col}(b_1, ..., b_n) \in \mathbb{R}^n$ such that, $b \ne 0$, $b_i \ge 0$, and $\lambda b_i = \sum_{j=1}^{n} \beta_{ij}b_j$, for $1 \le i \le n$. From this,

$$\lambda b_i\alpha_{ii} = \sum_{j\in J_i} \alpha_{ij}b_j, \qquad \text{for } i = 1, ..., n. \tag{1.5}$$

Let us write $I = \{i: b_i > 0\}$, $J = \{i: b_i = 0\}$ and notice that, $\lambda b_i\alpha_{ii} < b_i\alpha_{ii}$ if $i \in I$. Thus,

$$b_i\alpha_{ii} > \sum_{j\in K_i} \alpha_{ij}b_j, \tag{1.6}$$

where $K_i := \{j \in I: j \ne i\}$. In particular, there is nothing to prove if $J = \emptyset$.

Suppose now that $J \ne \emptyset$. Without loss of generality, we can write $I = \{1, ..., p\}$ for some $1 \le p < n$. Using (1.5), it is easy to prove that,

$$\alpha_{ij} = 0 \qquad \text{if } p < i \leq n, 1 \leq j \leq p. \tag{1.7}$$

Define matrices P, Q, R by $P = (\beta_{ij}\colon 1 \leq i, j \leq p)$, $Q = (\beta_{ij}\colon 1 \leq i \leq p;$ $P < j \leq n\}$, and $R = (\beta_{ij}\colon p < i, j \leq n\}$. Given $x \in \mathbb{C}^n$, let us write $x = \text{col}(x_I, x_J)$ with $x_I \in \mathbb{C}^p$ and $x_j \in \mathbb{C}^{n-p}$. Then, by (1.6), $Bx = \text{col}(Px_I + Qx_J, Rx_J)$ and thus, $\sigma(P) \subset \sigma(B)$, where $\sigma(P)$ denotes the set of all eigenvalues of P.

Claim. $\sigma(R) \subset \sigma(B)$. To show this, let us fix $\mu \in \sigma(R)$ and $w \in \mathbb{C}^{n-p}$, $w \neq 0$, such that, $R(w) = \mu w$. If $\mu \notin \sigma(B)$ then, $\mu \notin \sigma(P)$ and then, there exists $z \in \mathbb{C}^p$ such that, $(P - \mu I)z = -Qw$. If we put $x = \text{col}(z, w)$, then $Bx = \mu x$, and so, $\mu \in \sigma(A)$. This contradiction proves the claim.

By the above claim, $\nu(R) < 1$, and by induction, there exist $d_i > 0$, $i \in J$, such that,

$$d_i\alpha_{ii} > \sum_{j \in L_i} \alpha_{ij} d_j, \qquad \text{for } i \in J, \tag{1.8}$$

where $L_i := \{j \in J\colon j \neq i\}$. By (1.6), there exists $\varepsilon > 0$ such that,

$$\varepsilon \sum_{j>p} \alpha_{ij} d_j < b_i\alpha_{ii} - \sum_{j \in K_i} \alpha_{ij} b_j, \qquad \text{for all } i \in I. \tag{1.9}$$

Now, we shall show that (1.2) is satisfied if we define $c_i = b_i$ for $i \in I$ and $c_i = \varepsilon d_i$ for $i \in J$. If $i \in I$ then, by (1.9),

$$\sum_{j \in J_i} \alpha_{ij} c_j = \sum_{j \in K_i} \alpha_{ij} b_j + \varepsilon \sum_{j>p} \alpha_{ij} d_j < b_i\alpha_{ii} = c_i\alpha_{ii}.$$

If $i \in J$ then, by (1.7) and (1.8),

$$\sum_{j \in J_i} \alpha_{ij} c_j = \varepsilon \sum_{j \in L_i} \alpha_{ij} d_j < \varepsilon d_i\alpha_{ii} = c_i\alpha_{ii},$$

and the proof is complete.

1.2. Corollary. *The following conditions are equivalent*:

(a) $\nu(B) < 1$.

(b) $\nu(C) < 1$.

(c) *There exist positive constants* $c_1, \ldots, c_n$ *such that,*

$$c_i\alpha_{ii} > \sum_{j\in J_i} c_j\alpha_{ji}.$$

Proof. By Theorem 1.1 (with $A^* := (\alpha_{ji})$ in the place of A), we have that (b) and (c) are equivalent. Now, let us write $C^* = (\gamma_{ji})$ and let D be the diagonal matrix $\operatorname{diag}(\alpha_{11}, \ldots, \alpha_{nn})$. It is easy to show that $C^* = DBD^{-1}$, and the proof follows easily.

1.3. *Remark.* Let $Q = (q_{ij})$ be a 3×3 real matrix such that, $q_{ii} = 0$ for $i = 1, 2, 3$ and $q_{ij} \geq 0$ for all i, j. If $\det(I - Q) > 0$, then $\nu(Q) < 1$.

Proof. From the Perron–Frobenius theory [1], it suffices to show that the real eigenvalues of Q are less than one. To do this, let us write $q = q_{12}q_{21} + q_{13}q_{31} + q_{23}q_{32}$. We know that the real eigenvalues of Q are the real roots of the polynomial $p(s) = s^3 - qs - (q_{21}q_{23}q_{31} + q_{32}q_{21}q_{13})$. From this, $q < 1$, since $p(1) > 0$. On the other hand, $p'(s) = 3s^2 - q$ and thus, $p'(s) > 2$ for $s \geq 1$. The proof follows now easily.

2. A Continuation Result

In this section we prove Theorem 0.1. To this end, let us define a real $n \times n$ matrix function $B = (b_{ij})$ by:

(i) $b_{ii}(t) = 0$, for $i = 1, \ldots, n$ and $t \in \mathbb{R}$.

(ii) $b_{ij}(t) = |a_{ij}(t)|/a_{ii}(t)$ for $i \neq j$ and $t \in \mathbb{R}$.

The maximal domain of a solution u of (0.1) is denoted by $\operatorname{dom}(u)$. Notice that, if $u(\tau) > 0$ for some τ, then $u(t) > 0$ for all $t \in \operatorname{dom}(u)$. In this case, we say that u is positive.

Proof of Theorem 0.1. Suppose that there exists $\tau < T < +\infty$ such that, $\operatorname{dom}(u) \cap [\tau, +\infty) = [\tau, T)$.

By Theorem 1.1, there exist positive constants $c_1, \ldots, c_n$ such that,

$$c_i a_{ii}(T) > \sum_{j\in J_i} |a_{ij}(T)| c_j, \qquad \text{for } 1 \leq i \leq n;$$

and hence, there exists $S \in (\tau, T)$ and $m > 0$ such that,

$$c_i a_{ii}(t) \geq m + \sum_{j\in J_i} |a_{ij}(t)| c_j; \qquad S \leq t \leq T; 1 \leq i \leq n. \tag{2.1}$$

Let us define $M = \sup\{b_i(t)\colon S \le t \le T, 1 \le i \le n\}$ and fix a real number λ such that,

$$\lambda > \max\{M/m, u_1(S)/c_1, ..., u_n(S)/c_n\}. \tag{2.2}$$

CLAIM. *$u_i(t) < \lambda c_i$, for $S \le t < T$ and $1 \le i \le n$.*

Proof of the Claim. Assume that the claim is false. Then there exist a $\xi \in (S, T)$ and a $1 \le k \le n$ such that,

$$u_k(\xi) = \lambda c_{k'} u_k'(\xi) \ge 0, \qquad u_i(\xi) \le \lambda c_i \qquad \text{for } i = 1, ..., n.$$

From this and (0.1) and (2.1),

$$M \ge b_k(\xi) \ge \sum_{j=1}^{n} a_{kj}(\xi) u_j(\xi) = \lambda c_k a_{kk}(\xi) + \sum_{j \in J_k} a_{kj}(\xi) u_j(\xi)$$
$$\ge c_k a_{kk}(\xi) - \sum_{j \in J_k} |a_{kj}(\xi)| \lambda c_j \ge \lambda m.$$

This contradicts (2.2) and the proof of the claim is complete.

On the other hand, $u_1(t) + \cdots + u_n(t) \to +\infty$ as $t \to T^-$, and this contradiction ends the proof.

2.1. COROLLARY. *If $n = 2$ and $a_{11}(t)a_{22}(t) > |a_{12}(t)a_{21}(t)|$ for all $t \in \mathbb{R}$, then u is defined on $[\tau, \infty)$.*

From Remark 1.3 and Theorem 0.1, we obtain:

2.2. COROLLARY. *Assume $n = 3$. If $a_{11}a_{22}a_{33} > |a_{12}a_{21}|a_{33} + |a_{13}a_{31}| a_{22} + |a_{23}a_{32}|a_{11} + |a_{12}a_{23}a_{13}|$, then u is defined on $[\tau, \infty)$.*

3. THE MAIN RESULTS

In the following, we assume that a_{ij}, b_i are bounded and $u = (u_1, ..., u_n)$ denotes a positive solution to (0.1). We also fix $\tau \in \operatorname{dom}(u)$.

3.1. THEOREM. *If (0.2) holds for some positive constants m, c_1, ..., c_n, then u is defined and bounded on $[\tau, \infty)$.*

Proof. By Corollary 1.2 and Theorem 0.1, u is defined on $[\tau, \infty)$. Given $d = (d_1, ..., d_n) \in \mathbb{R}^n$, we define:

$$W_d(t) = \sum_{i=1}^{n} d_i \ln(u_i(t)), \qquad t \in [\tau, \infty). \tag{3.1}$$

Claim. W_c is bounded above on $[\tau, \infty)$, where $c = (c_1, ..., c_n)$. To show this, we remark that:

$$\begin{aligned}\sum_{i,j=1}^{n} c_i a_{ij}(t) u_j(t) &= \sum_{i=1}^{n} c_i a_{ii}(t) u_i(t) \\ &\quad + \sum_{i=1}^{n} \sum_{j \in J_i} c_j a_{ji}(t) u_i(t) \geq \sum_{i=1}^{n} c_i a_{ii}(t) u_i(t) \\ &\quad - \sum_{i=1}^{n} \sum_{j \in J_i} c_j |a_{ji}(t)| u_i(t) \geq m\|u(t)\| := m \sum_{i=1}^{n} u_i(t).\end{aligned}$$

From this, $W_c'(t) = \sum_{i=1}^n c_i b_i(t) - \sum_{i,j=1}^n c_i a_{ij}(t) u_j(t) \leq M\|c\| - m\|u(t)\|$, where $M := \sup\{b_i(t): t \geq \tau, 1 \leq i \leq n\}$.

Let us write $p = \max\{c_1, ..., c_n\}$. Using the concavity of ln, we have $W_c(t) \leq p\|u(t)\|$ and thus, $W_c'(t) + (m/p) W_c(t) \leq M\|c\|$, for $t \geq \tau$. The proof of the claim follows now easily.

Let us fix a positive constant K such that $\sum_{i=1}^n |a_{ji}(t)| \leq K$ for $1 \leq i \leq n$ and $t \in \mathbb{R}$, and fix $0 < \varepsilon < \min\{c_1, ..., c_n, m/4K\}$. Define N as the subset of $\mathbb{R}^n$ consisting of all points $d = (d_1, ..., d_n)$ such that $|d_i - c_i| < \varepsilon$ for $i = 1, ..., n$ and notice that, for such a d, we have $d > 0$ and

$$d_i a_{ii}(t) \geq m/2 + \sum_{j \in J_i} d_j |a_{ji}(t)|; \qquad 1 \leq i \leq n, t \in \mathbb{R}.$$

Since N is an open set, there exists a vector basis $c^1, ..., c^n$ of $\mathbb{R}^n$ such that $c^j \in N$ for $j = 1, ..., n$. If we write $c^j = (c_1^j, ..., c_n^j)$ then, $c_i^j > 0$ for all i, j and,

$$c_i^k a_{ii}(t) \geq (m/2) + \sum_{j \in J_i} c_j^k |a_{ji}(t)| \tag{3.2}$$

for $i, k = 1, ..., n$ and $t \in \mathbb{R}$.

Let us write $W_k = W_d$ for $d = c^k$, $W = (W_1, ..., W_n)$, $v_i(t) = \ln(u_i(t))$, and $v = (v_1, ..., v_n)$. By (3.1), $W(t) = Dv(t)$, where D is the $n \times n$ matrix

(c_i^k), and thus, $v(t) = D^{-1}W(t)$. By the above claim, $W(t)$ is bounded above $[\tau, T)$ and so, the same holds for $v(t)$. The proof is thereby complete.

From Theorems 1.2, 1.5, and 1.4 of [2], we obtain the following corollaries to Theorem 3.1.

3.2. COROLLARY. *Assume* (0.2) *holds. If* u, v *are positive solutions to* (0.1), *then* $u(t) - v(t) \to 0$ as $t \to +\infty$.

3.3. COROLLARY. *Assume* (0.2) *holds. Then, system* (0.1) *has at most one solution defined on* $\mathbb{R}$ *whose components are bounded above and below by positive constants.*

3.4. COROLLARY. *Assume* (0.2) *holds. If* a_{ij}, b_i *are* T*-periodic in the time* t *for all* i, j, *then there is a unique nonnegative* T*-periodic solution* u^0 *of* (0.1) *such that*, $u(t) - u^0(t) \to 0$ *as* $t \to +\infty$, *for any positive solution* u *of* (0.1).

3.5. COROLLARY. *Assume* (0.2) *holds. If* a_{ij}, b_i *are almost periodic for all* i, j *and* (0.1) *has a positive solution* $v = (v_1, \ldots, v_n)$ *such that*, $\liminf_{t\to+\infty} v_i(t) > 0$ *for all* i, *then* (0.1) *has an almost periodic solution whose components are bounded below by positive constants.*

Proof of Theorem 0.2. By Theorem 3.1, u is defined and bounded on $[\tau, \infty)$. Notice that a_{ij}, b_i are bounded for all i, j. Now, let us fix $\varepsilon > 0$ such that, $b_i(t)/a_{ii}(t) \geq \varepsilon$, for $t \in \mathbb{R}$ and $i = 1, \ldots, n$.

Claim. $u_i(t) \geq r_i := \min\{\varepsilon, u_i(\tau)\}$ for $t \geq \tau$. To show this, let us fix $0 < R_i < r_i$. We must prove that $u_i(t) > R_i$, for $t \geq \tau$. Suppose that this assertion is false. Since $u_i(\tau) > R_i$, there exists $s > \tau$ such that, $u_i(s) = R_i$ and $u_i'(s) \leq 0$. From this, $b_i(s) \leq a_{ii}(s)R_i < a_{ii}(s)r_i$, and so, $r_i > \varepsilon$. This contradiction proves the claim.

The proof follows now from Corollaries 3.5 and 3.2.

4. EXTINCTION

In this section we prove several results about the extinction of some species.

4.1. COROLLARY. *Suppose that* a_{ij}, b_i *are* T*-periodic for some* $T > 0$ *and assume that*,

$$\int_0^T b_i(s)\, ds < 0 \qquad \text{for } i = 1, \ldots, n. \tag{4.1}$$

If (0.2) holds then, $u(t) \to 0$ as $t \to +\infty$, for any positive solution u of (0.1).

Proof. By (4.1) we have that the trivial solution $w \equiv 0$ of (0.1) is locally asymptotically stable, and the proof follows from Corollary 3.4.

4.2. COROLLARY. *Assume a_{ij}, b_i are T-periodic and $b_1(s) > 0$, $s \in \mathbb{R}$. Suppose that* (0.2) *holds and let U be the unique positive and T-periodic solution of the logistic equation $x' = x[b_1(t) - a_{11}(t)x]$. If*

$$\int_0^T [b_i(s) - a_{i1}(s)U(s)]\, ds < 0 \qquad \text{for } 2 \le i \le n, \tag{4.2}$$

then $u(t) \to (U(t), 0, ..., 0)$ as $t \to +\infty$, for any positive solution u of (0.1).

Proof. Notice first that $(U(t), 0, ..., 0)$ is a nonnegative T-periodic solution of (0.1). Moreover, by (4.2) this solution is locally asymptotically stable and the proof follows as in Corollary 4.1.

4.3. PROPOSITION. *Assume that the hypothesis in Theorem* 3.1 *hold. If $b_i(t) \le 0$ for $1 \le i \le n$, then* $\min\{u_1(t), ..., u_n(t)\} \to 0$ *as $t \to +\infty$.*

Proof. Let us define $W(t) = \sum_{i=1}^n c_i \ln u_i((t))$. By the arguments in Theorem 3.1, we have $W'(t) \le -m\|u(t)\|$.

Claim. $W(t) \to -\infty$ as $t \to +\infty$. To show this suppose that $W(t)$ has a finite limit as $t \to +\infty$. Then, there exists a sequence $t_k \to +\infty$ such that $W'(t_k) \to 0$. From this, $\|u(t_k)\| \to 0$ and then, $W(t_k) \to -\infty$. This contradiction proves the claim.

Let us write $y(t) = \min\{u_i(t)\colon 1 \le i \le n\}$. Then, $W(t) \ge \sum_{i=1}^n c_i \ln(y(t))$, and the proof follows easily.

4.4. COROLLARY. *Suppose that the assumptions in Proposition* 4.3 *hold. If $n = 2$, then $u(t) \to 0$ as $t \to +\infty$.*

Proof. We know that,

$$\min\{u_1(t), u_2(t)\} \to 0 \qquad \text{as } t \to +\infty. \tag{4.3}$$

By Theorem 3.1, u is bounded on $[\tau, \infty)$ and thus we can define $q = \limsup_{t\to+\infty} u_1(t)$. By Corollary 1.2 of [3], there exists a sequence $t_k \to +\infty$ such that, $u_1'(t_k) \to 0$ and $u_1(t_k) \to q$ as $k \to +\infty$.

Assume now that $q > 0$. By (0.1),

$$b_1(t_k) - a_{11}(t)u_1(t_k) - a_{12}(t)u_2(t_k) \to 0 \qquad \text{as } k \to +\infty,$$

but, by (4.3), $u_2(t_k) \to 0$ and so, $a_{11}(t_k)u_1(t_k) \to 0$ as $k \to +\infty$, since $b_1 \le 0$. On the other hand, by (0.2), $\inf\{a_{11}(t)\colon t \in \mathbb{R}\} > 0$, and hence, $u_1(t_k) \to$

$0 = q$. This contradiction proves that $q = 0$, and thus, $u_1(t) \to 0$ as $t \to +\infty$. Analogously, $u_2(t) \to 0$ as $t \to +\infty$, and the proof is complete.

References

1. R. Bellman, "Introduction to Matrix Analysis," McGraw–Hill, New York, 1960.
2. A. Tineo, On the asymptotic behavior of some population models, *J. Math. Anal. Appl.* **167,** (1992), 516–529.
3. A Tineo, Asymptotic behavior of positive solutions to the nonautonomous Lotka–Volterra competition equations, *Differential Integral Equations* **6,** No. 2, (1993), 449–457.